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ADMISSIBLE REPRESENTATIONS OF EFFECTIVE CPO'S

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Abstract. The elements of a cpo (complete partial order) $\bar{D}$ are 'abstract' objects in general. A concrete machine cannot operate with abstract objects but only with names of objects. In this paper the set $\mathbb{F}$ of total functions from $\mathbb{N}$ to $\mathbb{N}$ is suggested as the set of names. 'Admissible representations' $\delta:\mathbb{F}\to D$ for effective cpo's are defined by two axioms, which generalize the axioms for Gödel numberings of the partial recursive functions. Topological and recursion theoretic considerations show that the definition is very natural. It is also proved that P_ω, the set of subsets of $\mathbb{N}$, is not as suitable as a set of names.

Finally it is proved that for admissible representations the computable functions over cpo's are represented by the computable extensional functions over the names.

1. Introduction

Effective complete partial orders are a useful tool for a generalized study of computable functions and computable operators (Scott [15], Egli and Constable [3], Plotkin [10], Smyth [17], Sciore and Tang [14], Kanda and Park [7], Weihrauch and Deil [19] and Weihrauch [18]). The computable elements of an effective cpo can be numbered admissibly [3, 14, 7, 19]. Computability, then, must finally be definable by computable extensional functions on the numbers. One important application of this general theory is computable analysis. For this purpose also computability on non-denumerable sets (on all real numbers) should be definable. Thus, the concept of numbering is no longer useful. A non-denumerable standard system of names, on which computability can be defined, is needed. Classical Recursion Theory suggests at least two different possibilities: P_ω (the subsets of $\mathbb{N}$) together with the 'enumeration operators' and $\mathbb{F}$ (the total functions over $\mathbb{N}$) together with the 'general recursive operators' (see Rogers [12]). In computable analysis, $\mathbb{F}$ (the total functions over $\mathbb{N}$) was used as set of names (see, e.g., Hauck [6]). In this report we generalize this principle and study representations $\delta:\mathbb{F}\to D$, where D is an effective cpo.

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For providing a simple concrete basis for subsequent proofs a definition of computability on $\mathbb{F}$ is given in Section 3. In Section 4 a natural representation ('standard representation') of an effective cpo is constructed, and 'admissible' representations are defined by two axioms which characterize them as computably m-equivalent to a standard representation. The two axioms have very obvious and natural meanings and generalize the axioms for acceptable numberings of the partial recursive functions (Rogers [13]) and for admissible numberings of the recursively enumerable elements of an effective cpo (Weihrauch and Deil [19]). Admissible representations are studied under recursion theoretical and topological point of view. A single-valuedness operator is the tool for proving the main results. Finally it is proved that for effective representations by P_ω instead of $\mathbb{F}$ a single-valuedness operator cannot exist in general. This strongly justifies the choice of $\mathbb{F}$.

In Section 5 the connection between the continuous (computable) functions of $\mathbb{F}$ and the continuous (computable) functions between cpo's, induced by admissible representations, is studied. We claim that for defining actual computations on cpo's the approach using admissible representations $\delta : \mathbb{F} \to D$ is the most natural one.

The following notations will be used.

$\mathbb{N}^*$	:= set of all (finite) words over $\mathbb{N}$,
ν_W	:= some standard numbering of $\mathbb{N}^*$,
$\lg(w)$	:= length of $w \in \mathbb{N}^*$,
$R^{(1)}$	:= class of all unary total recursive functions,
$P^{(1)}$	:= class of all unary partial recursive functions,
φ	:= standard numbering of $P^{(1)}$,
W_i	:= domain of φ_i,
$\langle\ \rangle : \mathbb{N}^2 \to \mathbb{N}$	:= standard pairing function with inverse $(\pi_1, \pi_2) : \mathbb{N} \to \mathbb{N}^2$.

2. The basic definitions and concepts

We summarize in short the basic definitions which will be used in this report. For a more detailed discussion, see Weihrauch and Deil [19], Egli and Constable [3] and Smyth [17].

Let $\bar{D} = (D, \sqsubseteq)$ be a partial order. $A \subseteq D$ is directed iff $A \neq \emptyset$ and $(\forall a, b \in A)(\exists c \in A)\, a, b \sqsubseteq c$. $\bar{D}$ is complete iff $\bigsqcup A := \sup A$ exists for every directed subset $A \subseteq D$. A cpo is a tuple $\bar{D} = (D, \sqsubseteq, \perp)$ where $(D, \sqsubseteq)$ is a complete partial order with $\perp \in D$ as its minimum.

On D a binary relation $\prec$ (Scott [15]) is defined as follows: $x \prec y$ iff $y \sqsubseteq \bigsqcup A \Rightarrow (\exists a \in A)\, x \sqsubseteq a$ for every directed subset $A \subseteq D$. Note, that $\prec$ is transitive. A subset $B \subseteq D$ is a basis for the cpo $\bar{D}$, iff $B_x := \{b \in B \mid b \prec x\}$ is directed and $x = \bigsqcup B_x$ for every $x \in D$. The cpo $\bar{D}$ is continuous iff there exists a basis for $\bar{D}$.

Examples for illustrating these definitions can be found in most of the already mentioned papers on cpo's (see, e.g., [14–19]). There is a useful lemma originally given by Scott.

Lemma 1. *Let $\bar{D}$ be a (continuous) cpo with basis B. Then the following statements hold*:

(1) $\bot \in B$,

(2) $(\forall x, y \in D)(\exists a \in B)(x \ll y \Rightarrow x \ll a \ll y)$,

(3) *B_x is $\ll$-directed for every $x \in D$,*

(4) *let $X \subseteq D$ be directed, then $y \ll \bigsqcup X \Leftrightarrow (\exists x \in X) y \ll x$.*

For a proof, see [5, 15] or [19].

Corollary 2. *Let $\bar{D}$ be a cpo with basis B, let $X \subseteq D$ be directed. Then $M := \bigcup\{B_x \mid x \in X\} = \{b \in B \mid b \ll \bigsqcup X\}$* and *$\bigsqcup X = \bigsqcup M$.*

Let $\bar{D}_1, \bar{D}_2$ be cpo's. A function $f: D_1 \to D_2$ is $(\bar{D}_1, \bar{D}_2)$-continuous (shortly: continuous, iff f is monotone and $f \bigsqcup X = \bigsqcup fX$ for every directed $X \subseteq D_1$.

For continuous cpo's $\bar{D}$, a canonical topology can be defined (Scott [15]). A subset $X \subseteq D$ is open, iff axioms (O1) and (O2) hold.

(O1) $(x \in X$ and $x \sqsubseteq y) \Rightarrow y \in X$

(O2) $(\forall x \in X)(\exists y \in X)\ y \ll x$.

Let O_D be the set of all open subsets of D. Then $\tau_D = (D, O_D)$ is a topology. If B is a basis for $\bar{D}$, then $\tilde{B} := \{O_b \mid b \in B\}$, where $O_b = \{x \mid b \ll x\}$ is a base of the topology τ_D. For continuous cpo's $\bar{D}_1$ and $\bar{D}_2$, a function $f: D_1 \to D_2$ is $(\bar{D}_1, \bar{D}_2)$-continuous, iff it is (τ_{D_1}, τ_{D_2})-continuous.

Essential for this paper is the way computability is introduced.

Main Definition. An *effective cpo* is a quadruple $\bar{D} = (D, \sqsubseteq, \bot, \beta)$, where $(D, \sqsubseteq, \bot)$ is a continuous cpo and $\beta: \mathbb{N} \to B$ is a numbering of a basis of $(D, \sqsubseteq, \bot)$ such that the effectivity axiom (E) holds:

(E) $\{(i, j) \mid \beta(i) \ll \beta(j)\}$ is recursively enumerable.

For effective cpo's there is a very natural theory of computability [18, 19]. Axiom (E) is similar to approaches used by Sciore and Tang [14] or Smyth [17], however, it seems to be the most simple and natural assumption. Constable and Egli [3] and Kanda and Park [7], for example, consider algebraic cpo's and recursiveness of $\{(i, j) \mid \beta(i) \ll \beta(j)\}$.

Now computability can be defined.

Definition 3. For an effective cpo $\bar{D}$, $x \in D$ is recursively enumerable (r.e.) iff $\{i \in \mathbb{N} \mid \beta(i) \ll x\}$ is recursively enumerable.

For effective cpo's $\bar{D}_1$ and $\bar{D}_2$, a function $f:D_1\to D_2$ is $(\bar{D}_1,\bar{D}_2)$-computable (shortly: computable) iff f is $(\bar{D}_1,\bar{D}_2)$-continuous and

$$\{(i,j)\mid \beta_2(j)\prec f\beta_1(i)\}\quad \text{is recursively enumerable.}$$

Effective cpo's and computability of this kind are studied in detail in [18] and [19].

As a generalization of the 'acceptable numberings' of the partial recursive functions admissible numberings of the r.e. elements of effective cpo's are defined.

Definition 4. Let D_{re} be the set of all r.e. elements of an effective cpo $\bar{D}$. $\mu:\mathbb{N}\to D_{re}$ (surjective) is an admissible numbering of D_{re} iff (Z1) and (Z2) hold:

(Z1) $\{\langle i,j\rangle\mid \beta(i)\prec\mu(j)\}$ is recursively enumerable.

(Z2) There is a function $f\in R^{(1)}$ with

$(\forall i\in\mathbb{N})(\beta W_i \text{ directed } \Rightarrow \mu f(i)=\bigsqcup\beta W_i)$.

For a general theory of numberings see Eršov [4] or Mal'cev [9].

Admissible numberings of D_{re} exist for any effective cpo $\bar{D}$ (see [19, 18]), and examples for admissible numberings of the r.e. elements in special cpo's include well-known standard numberings, for example the numbering W of the recursively enumerable sets [12], the Gödelnumberings φ of the partial recurive functions etc. (these exampies can be found in [19]).

A generalization of Roger's Isomorphism Theorem for Gödelnumbering holds in the case of admissible numberings: It is well known, that for precomplete numberings m-equivalence induces isomorphism (Eršov [4]), and in [18, 19] it is shown that the admissible numberings of effective cpo's are precomplete.

3. Continuous and computable functions on $\mathbb{F}$

As we have already mentioned, we choose $\mathbb{F}=\mathbb{N}^{\mathbb{N}}=\{f:\mathbb{N}\to\mathbb{N}\}$ as a non-denumerable standard system of names for the elements of an effective cpo. For this purpose it is reasonable to have an explicit definition of computability on $\mathbb{F}$. Such a definition is given in this section.

For $f\in\mathbb{F}$ and $n\in\mathbb{N}$ let $f^{[n]}:=f(0)f(1)\cdots f(n)\in\mathbb{N}^*$ and for $w\in\mathbb{N}^*$ define $[w]:=\{f\in\mathbb{F}\mid w=f(0)\cdots f(n-1)\}$, where $n=\lg(w)$. There is a well-known topology on $\mathbb{F}$, Baire's Topology τ_B, which is defined by he topological base

$$B=\{[w]\mid w\in\mathbb{N}^*\}.$$

On $\mathbb{N}^*$ we consider the prefix ordering $x\sqsubseteq y :\Leftrightarrow x$ is prefix of y.

Definition 5. Let $\gamma:\mathbb{N}^*\to\mathbb{N}^*$, $h\in\mathbb{F}$ and $A\subseteq\mathbb{F}$.

- γ is monotone on h, iff $(x\sqsubseteq y \text{ and } h\in[y])\Rightarrow\gamma(x)\sqsubseteq\gamma(y)$.
- γ is unbounded on h, iff $(\forall n)(\exists y)(h\in[y] \text{ and } \lg\gamma(y)>n)$.

- γ is monotone on A, iff γ is monotone on g for all $g \in A$.
- γ is unbounded on A, iff γ is unbounded on g for all $g \in A$.

There is a direct connection between functions $\gamma:\mathbb{N}^* \to \mathbb{N}^*$ and certain continuous functions on Baire's space.

Theorem 6. (1) *Let $\gamma:\mathbb{N}^* \to \mathbb{N}^*$ be a function. Then γ uniquely determines a τ_B-continuous function $\Gamma: A \to \mathbb{F}$ with*

$$A = \{f \in \mathbb{F} \mid \gamma \text{ is monotone and unbounded on } f\}$$

and

$$f \in [w] \Rightarrow \Gamma f \in [\gamma w] \quad \text{for all } f \in A \text{ and } w \in \mathbb{N}^*.$$

(2) *Let $\Gamma: A \to \mathbb{F}$, $A \subseteq \mathbb{F}$, be τ_B-continuous. Then there is some $\gamma:\mathbb{N}^* \to \mathbb{N}^*$ with*

$$\gamma \text{ is monotone and unbounded on } f \quad \text{for all } f \in A,$$

and

$$f \in [w] \Rightarrow \Gamma f \in [\gamma w] \quad \text{for all } f \in A \text{ and } w \in \mathbb{N}^*.$$

Proof. (1) Let γ be given, define A as above. Let $\mathrm{Def}(\Gamma) := A$ and for $f \in A$ define Γf by

$$\{\Gamma f\} := \bigcap_i [\gamma f^{[i]}].$$

Since $f^{[i]} \sqsubseteq f^{[i+1]}$ and γ is monotone and unbounded on f, Γf is well-defined. For $y \in \mathbb{N}^*$ we have

$$\begin{aligned} f \in \Gamma^{-1}[y] &\Leftrightarrow f \in A \text{ and } \Gamma f \in [y] \\ &\Leftrightarrow f \in A \text{ and } \textstyle\bigcap_i [\gamma f^{[i]}] \subseteq [y] \\ &\Leftrightarrow f \in A \text{ and } (\exists i)[\gamma f^{[i]}] \subseteq [y] \\ &\Leftrightarrow (\exists w)(f \in [w] \text{ and } [\gamma w] \subseteq [y]) \\ &\Leftrightarrow f \in A \text{ and } f \in \textstyle\bigcup\{[w] \mid [\gamma w] \subseteq [y]\}. \end{aligned}$$

Therefore, $\Gamma^{-1}[y] = A \cap \bigcup\{[w] \mid [\gamma w] \subseteq [y]\}$, i.e., $\Gamma^{-1}[y]$ is open w.r.t. the relative topology on A, and Γ is continuous. Γ is unique since γ is unbounded on A.

(2) Let $\Gamma: A \to \mathbb{F}$ be continuous. Then, by continuity, $\Gamma f \in [y] \Rightarrow (\exists w)(f \in [w]$ and $\Gamma[w] \subseteq [y])$ for all $f \in A$. For $w \in \mathbb{N}^*$ define

$$M_w := \{y \in \mathbb{N}^* \mid \lg(y) \le \lg(w) \text{ and } \Gamma[w] \subseteq [y]\}.$$

If $[y_1] \cap [y_2] \neq \emptyset$, then $y_1 \sqsubseteq y_2$ or $y_2 \sqsubseteq y_1$. Therefore, $\max M_w$ exists, whenever $\Gamma[w] \neq \emptyset$ (note that $\varepsilon \in M_w$). Define γ by

$$\gamma(w) := \begin{cases} \max M_w & \text{if } A \cap [w] \neq \emptyset, \\ \varepsilon & \text{otherwise.} \end{cases}$$

Then γ is well-defined. γ is monotone on f for every $f \in A$. We prove that γ is unbounded on A. Suppose, $f \in A$. Then

$$
\begin{aligned}
\Gamma f \in [y] &\Rightarrow (\exists w)(f \in [w] \wedge \Gamma[w] \subseteq [y]) \quad \text{(by continuity of } \Gamma) \\
&\Rightarrow (\exists w)(\lg(y) \leq \lg(w) \wedge f \in [w] \wedge \Gamma[w] \subseteq [y]) \\
&\Rightarrow (\exists w)(f \in [w] \wedge y \in M_w) \\
&\Rightarrow (\exists w)(f \in [w] \wedge y \sqsubseteq \gamma(w)) \quad \text{for all } y \in \mathbb{N}^*.
\end{aligned}
$$

Since $f \in \mathrm{Def}(\Gamma)$ there are arbitrarily long y with $\Gamma f \in [y]$. Therefore, γ is unbounded on f. Finally, $f \in [w] \Rightarrow \Gamma f \in \Gamma[w] \subseteq [\gamma w]$ by definition of γ. $\square$

Thus, there is a correspondence between functions on $\mathbb{N}^*$ and continuous functions on $\mathbb{F}$.

Observe, that computability on $\mathbb{N}^*$ can be defined canonically by using the one-one surjective coding $\tau^* : \mathbb{N}^* \to \mathbb{N}$ (Rogers [12]), i.e., using an 'effective' numbering such as $(\tau^*)^{-1}$ for $\mathbb{N}^*$ (Reiser and Weihrauch [11]). Computability of $\mathbb{N}^*$ has been extensively studied, see e.g., Asser [1].

Computable functions on $\mathbb{F}$ are now defined by computable functions on $\mathbb{N}^*$.

Definition 7. For $\gamma : \mathbb{N}^* \to \mathbb{N}^*$ let $\psi(\gamma)$ be the function Γ from Theorem 6(1). Let $A \subseteq \mathbb{F}$ and $\Gamma : A \to \mathbb{F}$. Γ is computable, iff $\Gamma = \psi(\gamma)$ for some computable $\gamma : \mathbb{N}^* \to \mathbb{N}^*$. Define $C := \{\Gamma : \mathbb{F} \to \mathbb{F} \mid \Gamma \text{ computable}\}$. If $\Gamma = \psi(\gamma)$, then γ gives a way to approximate $\Gamma(f)$ by finite approximate values of f for any given $f \in \mathbb{F}$.

This definition is quite similar to that of computable functionals given by Davis [2], and $\Gamma \in C$, iff Γ is the restriction to $\mathbb{F}$ of a 'general recursive operator' (Rogers [12]). As a simple fact we note, that Γf is total-recursive, whenever Γ is computable and $f \in \mathrm{Dom}(\Gamma)$ is total-recursive.

4. Admissible representations

Let $\bar{D} = (D, \sqsubseteq, \perp, \beta)$ be an effective cpo. By continuity, for any $x \in D$ there is some $A \subseteq \mathbb{N}$ such that $\beta(A)$ is directed and $x = \bigsqcup \beta(A)$. Let $I_x := \{i \mid \beta(i) < x\}$, then I_x can be considered as a standard name of $x \in D$. A very natural representation $\bar{\delta} : P_\omega \dashrightarrow D$ can therefore be defined by

$$\mathrm{Dom}(\bar{\delta}) := \{I_x \mid x \in D\}, \qquad \bar{\delta}(I_x) := \bigsqcup \beta I_x.$$

Another possibility is to choose enumeration functions instead of sets as names. Let

$$\mathrm{MON} := \{f \in \mathbb{F} \mid (\forall i) \beta f(i) < \beta f(i+1)\}.$$

A natural representation $\delta_m : \mathbb{F} \dashrightarrow D$ can be defined by $\mathrm{Dom}(\delta_m) := \mathrm{MON}$, $\delta_m(f) := \bigsqcup \{\beta f(i) \mid i \in \mathbb{N}\}$. (Note that δ_m is onto D.) We shall prove that δ_m can be extended to a (total) representation $\delta_s:\mathbb{F} \dashrightarrow D$ of D. The main tool for proving this is a computable single-valuedness operator Γ_s (see Rogers [12]). Egli and Constable [3] use a similar construction to define their 'computable operators of type τ' by compatible sets. In our more general concept of effective cpo's we can only make use of the effectiveness axiom (E). Especially, it is not decidable whether a finite set of basic elements has a lub, and therefore the construction of Γ_s is more complicated in our case. We shall define 'admissible representations' by two axioms which generalize the universal Turing machine theorem and the smn-theorem of the theory of recursive functions (Rogers [13]). We shall show that these axioms characterize δ_s up to computable equivalence. Although $\bar{\delta}:P_\omega \dashrightarrow D$ might seem to be more natural, a similar theory does not exist in this case. We shall prove that for certain cpo's, $\bar{\delta}$ cannot be equivalent to some total representation $\delta:P_\omega \to D$.

In the following we shall assume, that $\bar{D} = (D, \sqsubseteq, \perp, \beta)$ is an effective cpo.

Theorem 8. *There exists a* $\Gamma_s:\mathbb{F}\to\mathbb{F}$, Γ_s *computable, with properties* (1) *and* (2) *for any* $f\in\mathbb{F}$:

(1) $(\forall i)\beta(\Gamma_s f)(i) < \beta(\Gamma_s f)(i+1)$,

(2) β range(f) *directed* $\Rightarrow \bigsqcup \beta$ range$(f) = \bigsqcup_i \beta(\Gamma_s f)(i)$.

Proof. We define a monotone and unbounded $\gamma:\mathbb{N}^*\to\mathbb{N}^*$, such that $\psi(\gamma):\mathbb{F}\to\mathbb{F}$ has the desired properties. Since $\{\langle i,j\rangle \mid \beta(i)<\beta(j)\}$ is r.e. and $\perp\in B$, there is an $h\in R^{(2)}$ with $\{h(i,j) \mid i\in\mathbb{N}\} = \{i \mid \beta(i)<\beta(j)\}$. Furthermore, there is some $g\in R^{(1)}$ with $\{\langle i,j\rangle \mid \beta(i)<\beta(j)\} = $ range g.

We define functions $\gamma:\mathbb{N}^*\to\mathbb{N}^*$ and $p, r:\mathbb{N}^*\to\mathbb{N}$ inductively as follows:

$$\gamma(x) := i_\perp \in\mathbb{N}^*, \qquad r(x) := 1, \qquad p(x) := i_\perp$$

if $\lg(x)\le 1$, where $\beta(i_\perp) = \perp$.

Suppose, $x = x_0x_1\cdots x_n$ with $x_i\in\mathbb{N}$ for $0\le i\le n$. Suppose, $\gamma(x) := ya$, where $y\in\mathbb{N}^*$ and $a\in\mathbb{N}$, $r(x)$ and $p(x)$ have already been defined. Let $r(x) =: \langle k, m\rangle$. For $e\in\mathbb{N}$, define $\gamma(xe)$, $p(xe)$, $r(xe)$ as follows. If there is some $\langle i,j\rangle\le n$ with

$$\{\langle h(m,x_k), h(j,x_i)\rangle, \langle p(x), h(j,x_i)\rangle\} \subseteq \{g(0),\ldots,g(n)\},$$

then

$$\gamma(xe) := \gamma(x)p(x),$$

$$p(xe) := h(j,x_i) \quad \text{for the first number } \langle i,j\rangle \text{ with this property,}$$

$$r(xe) := r(x)+1.$$

If no such number $\langle i,j\rangle$ exists, then

$$d := \pi_1\mu\langle c,t\rangle[\{\langle a,c\rangle,\langle c,p(x)\rangle\} \subseteq \{g(0),\ldots,g(t)\}],$$

$$\gamma(xe) := \gamma(x)d, \qquad p(xe) := p(x), \qquad r(xe) := r(x).$$

We shall prove now that $\gamma:\mathbb{N}^*\to\mathbb{N}^*$ exists and that $\psi(\gamma):\mathbb{F}\to\mathbb{F}$ has the desired properties.

(a) γ is computable. The number d defined by minimalization always exists, since the relation $<$ is dense on the basis (Lemma 1(2)).

(b) γ is monotone and unbounded by definition. Therefore $\psi(\gamma) =: \Gamma_s \in C$.

(c) An easy proof by induction yields $(\gamma(x) = ya \Rightarrow a < p(x))$. Therefore, $(\forall i)$ $(\Gamma_s f)(i) < (\Gamma_s f)(i+1)$, i.e., property (1) holds.

Suppose now that β range(f) is directed.

(d) The sequence $(r(f^{[n]}))_{n\in\mathbb{N}}$ is not finally constant. Let $\langle k, m\rangle := r(f^{[n]})$. By Corollary 2 there are $\langle i,j\rangle$ and n' with $\langle i,j\rangle \leq n'$ such that

$$\{\langle h(m,f(k)), h(j,f(i))\rangle, \langle p(f^{[n]}), h(j,f(i))\rangle\}\subseteq$$
$$\subseteq \{g(0), g(1), \ldots, g(n')\}.$$

Therefore, $r(f^{[n'+1]}) > r(f^{[n]})$.

(e) Suppose $b\in B$ and $b < \beta f(k)$. Then there is some m with $\beta h(m,f(k)) = b$. Since r is not finally constant, $b < \beta(\Gamma_s f)(n')$ for some $n'\in\mathbb{N}$. This implies $\bigsqcup \beta$ range$(f) \sqsubseteq \bigsqcup_i \beta(\Gamma_s f)(i)$.

(f) For all n, $\beta(\Gamma_s f)(n) < \beta f(i)$ for some i. Therefore $\bigsqcup_i \beta(\Gamma_s f)(i) \sqsubseteq \bigsqcup \beta$ range(f).

This proves property (2). □

Using Γ_s we shall define the standard representation of D.

Definition 9. Define

$$\mathrm{MON} := \{f \in \mathbb{F} \mid (\forall i)\beta f(i) < \beta f(i+1)\}$$

and

$$\delta_m : \mathrm{MON} \to D \text{ by } \delta_m(f) := \bigsqcup_i \beta f(i).$$

(Note that range $(\delta_m) = D$ by Theorem 8.) A function $\delta_s:\mathbb{F}\to D$ is called a *standard representation* of D, iff $\delta_s = \delta_m\Gamma_s$, where Γ_s is from Theorem 8.

The partial representation δ_m has interesting topological properties which will be used below.

Lemma 10. *δ_m is (τ', τ_D)-continuous and (τ', τ_D)-open, where τ' is the topology induced by τ_B on* MON.

Proof. Suppose $O \subseteq D$, O is τ_D-open. Consider $f\in M$ with $\delta_m f\in O$. By Axiom (O2), $y < \delta_m f$ for some $y\in O$, therefore $y < \beta f(i)$ for some i by Lemma 1(4). But then

$$g\in \mathrm{MON}\cap[f^{[i]}] \Rightarrow y < \beta f(i) < \delta g \Rightarrow \delta g\in O \quad \text{by Axiom (O1).}$$

Therefore, $\delta_m^{-1}O$ is the union of τ'-open sets, and δ_m is continuous. Suppose $w \in \mathbb{N}^*$ and $a \in \mathbb{N}$. If $[wa] \cap \mathrm{MON} = \emptyset$, then $\delta_m([wa] \cap \mathrm{MON})$ is open. Otherwise $\delta_m([wa] \cap \mathrm{MON}) = \{x \in D \mid \beta(a) < x\}$, i.e., an open set. Therefore $\delta_m A$ is τ_D-open whenever A is τ'-open. □

Our next aim is to characterize the standard representations by effectiveness axioms. The axioms have similar interpretations as the universal Turing Machine theorem and the smn-theorem of ordinary recursion theory, which already characterize uniquely (up to isomorphism) the Gödelnumberings of the partial recursive functions (Rogers [13]).

Definition 11. A representation $\delta : \mathbb{F} \to D$ (onto) of an effective cpo $\bar{D}$ is *admissible*, iff (A1) and (A2) hold:

(A1) There is a function $\Delta \in C$ with $(\forall f \in \mathbb{F})$ range $\Delta(f) = \{i \mid \beta(i) < \delta(f)\}$.

(A2) There is a function $\Sigma \in C$ with $\delta\Sigma f = \bigsqcup \beta$ range(f), whenever β range(f) is directed.

Thus we call a representation $\delta : \mathbb{F} \to D$ admissible, if the components of an element can be determined uniformly effectively from its name and if a name can be determined from an enumeration of components. (A1) is the axiom of effective decomposition, (A2) is the axiom of effective synthesis, corresponding to the utm-theorem and the smn-theorem, respectively. Equivalent to (A1) is the following condition: there is a 'test function' $\Delta' \in C$ with $\beta(i) < \delta f \Leftrightarrow \Delta'(i \oplus f) \neq \lambda x.0$, where

$$i \oplus f(j) = \begin{cases} i & \text{for } j = 0, \\ f(j-1) & \text{otherwise.} \end{cases}$$

We have to show, that admissible representations exist.

Theorem 12. *Let $\bar{D}$ be an effective cpo. Every standard representation is admissible.*

Proof. (A2) holds with $\Sigma(f) = f$.

To prove (A1), let $\gamma : \mathbb{N}^* \to \mathbb{N}^*$ be a function for Γ_s satisfying Theorem 6(2). Define $\delta : \mathbb{N}^* \to \mathbb{N}^*$ by

$$\delta(\varepsilon) := 0$$

and for $w = iw'$, $a \in \mathbb{N}$,

$$\delta(wa) := \begin{cases} \delta(w)1 & \text{if} (\exists b)(b \text{ is some symbol from } \gamma(w) \text{ and} \\ & \langle i, b\rangle \in \{g(0), \ldots, (\lg(w))\}), \\ \delta(w)0 & \text{otherwise} \end{cases}$$

where $g \in R^{(1)}$ with range $(g) = \{\langle i, j\rangle \mid \beta(i) < \beta(j)\}$.

Then δ is computable, monotone and unbounded. Let Δ be defined by δ via Theorem 6(1).

Then $\beta(i) < \delta_s f \Leftrightarrow \Delta(i \oplus f) \neq \lambda x.0$. □

In order to discuss the meaning of our 'effectiveness axioms' we define reducibility and equivalence for representations in analogy to m-reducibility for numberings.

Definition 13. For $\delta_i : A_i \to D$ with $A_i \subseteq \mathbb{F}$ $(i = 1, 2)$ we define

$$\delta_1 \leq \delta_2 \;:\Leftrightarrow\; (\exists \Omega \in C)(\forall f \in A_1)\delta_1 f = \delta_2 \Omega f,$$

$$\delta_1 \equiv \delta_2 \;:\Leftrightarrow\; (\delta_1 \leq \delta_2 \text{ and } \delta_2 \leq \delta_1).$$

The next lemma relates axiom (A1) and axiom (A2).

Lemma 14. *Let $\bar{D}$ be an effective cpo, $\delta : \mathbb{F} \to D$.*
(1) *δ satisfies axiom* (A1) $\Leftrightarrow$ $\delta \leq \delta_{\mathrm{m}}$,
(2) *δ satisfies axiom* (A2) $\Leftrightarrow$ $\delta_{\mathrm{m}} \leq \delta$,
where δ_{m} is from Definition 9.

Proof. (1) ($\Rightarrow$) By assumption, there is a $\Gamma \in C$ with range $\Gamma(f) = \{i \mid \beta(i) < \delta f\}$. Thus, β range $\Gamma(f)$ is a directed set in D and therefore $\delta f = \bigsqcup \beta \Gamma f(i) = \bigsqcup \beta \Gamma_{\mathrm{s}} \Gamma f(i) = \delta_{\mathrm{m}} \Gamma_{\mathrm{s}} \Gamma(f)$ with $\Gamma_{\mathrm{s}} \circ \Gamma \in C$, that is $\delta \leq \delta_{\mathrm{m}}$.

($\Leftarrow$) Let $\delta = \delta_{\mathrm{m}} \Omega$, then

$$\beta(i) < \delta f \;\Leftrightarrow\; \beta(i) < \delta_{\mathrm{m}} \Omega(f) \;\Leftrightarrow\; \beta(i) < \delta_{\mathrm{s}} \Omega(f) \Leftrightarrow i \in \text{range } \Delta(\Omega(f))$$

where $\Delta \in C$ is the function from (A1) for δ_{s} (see Theorem 12). Hence, $\Delta\Omega \in C$ is the function claimed in (A1) for δ.

(2) ($\Rightarrow$) For $f \in \mathrm{MON}$, $\delta_{\mathrm{m}}(f) = \bigsqcup \beta f(i) = \bigsqcup \beta$ range $(f) = \delta \Sigma f$, where $\Sigma \in C$ is the function from (A2) for δ, therefore we have $\delta_{\mathrm{m}} \leq \delta$.

($\Leftarrow$) Suppose $\delta_{\mathrm{m}} = \delta \Omega$ for $\Omega \in C$ and let β range(f) be directed. Then

$$\bigsqcup \beta \text{ range}(f) = \bigsqcup \beta \text{ range } \Gamma_{\mathrm{s}}(f) = \delta_{\mathrm{m}} \Gamma_{\mathrm{s}}(f) = \delta \Omega \Gamma_{\mathrm{s}}(f),$$

so $\Omega \circ \Gamma_{\mathrm{s}} \in C$ is the function satisfying (A2) for δ. □

Thus we get the following characterizations of the axioms (A1) and (A2), where δ_{m} and δ_{s} are as in Definition 9.

Corollary 15. *Let $\bar{D}$ be an effective cpo, $\delta : \mathbb{F} \to D$. Then the following statements are equivalent*:
(1) *δ satisfies axiom* (A1),
(2) $\delta \leq \delta_{\mathrm{m}}$,
(3) $\delta \leq \delta_{\mathrm{s}}$,
(4) *$\delta \leq \delta'$ for all $\delta' : \mathbb{F} \to D$ satisfying* (A2).

Proof. (1) $\Leftrightarrow$ (2): Follows by Lemma 14(1).

(2) $\Leftrightarrow$ (3): By definition of a standard representation δ_{s} and the properties of Γ_{s} we get $\delta_{\mathrm{m}} \equiv \delta_{\mathrm{s}}$.

(3) $\Leftrightarrow$ (4): Follows by Lemma 14(1), (2). □

Corollary 16. *Let $\bar{D}$ be an effective cpo, $\delta:\mathbb{F}\to D$. Then the following statements are equivalent*:

(1) *δ satisfies* (A2),

(2) *$\delta_m \leq \delta$,*

(3) *$\delta_s \leq \delta$,*

(4) *$\delta' \leq \delta$ for all $\delta':\mathbb{F}\to D$ satisfying* (A1).

Proof. (1) $\Leftrightarrow$ (2): Follows by Lemma 14(2).

(2) $\Leftrightarrow$ (3): Follows since $\delta_m \equiv \delta_s$.

(2) $\Leftrightarrow$ (4): Follows by Lemma 14(1), (2). □

The characterization theorem for admissible representations immediately follows.

Theorem 17. *Let δ be a representation for an effective cpo $\bar{D}$. Then the following properties are equivalent*:

(1) *δ is admissible.*

(2) *δ is a maximum (up to equivalence) in the class of all representations for $\bar{D}$ satisfying* (A1).

(3) *δ is a minimum (up to equivalence) in the class of all representations for $\bar{D}$ satisfying* (A2).

(4) *$\delta \equiv \delta_s$ for some standard representation δ_s for $\bar{D}$.*

Note that it is not true in general that $\delta \equiv \delta_s$ implies 1-1-equivalence or even isomorphism. Let $D := \{\bot, \top\}$, $\bot \sqsubseteq \top$, $\beta(0) := \bot$, $\beta(n+1) := \top$. Then δ_1 and δ_2 with $\delta_1(f) := \bigsqcup_i \beta f(i)$, and $\delta_2(f) := \bigsqcup_i \beta\lfloor\frac{1}{2}f(i)\rfloor$ are admissible, but $\delta_1^{-1}(\bot) = \lambda x.0$ and $\delta_2^{-1}(\bot) = \{0, 1\}^{\mathbb{N}}$. Therefore, there cannot exist an injective translation from δ_2 to δ_1.

There is a close connection between admissible representations $\delta:\mathbb{F}\to D$ and admissible numberings $\eta:\mathbb{N}\to D_{re}$ (see [18, 19]).

Theorem 18. *Let δ be an admissible representation of an effective cpo $\bar{D}$, φ a standard numbering of $P^{(1)}$. Then $\delta\circ\varphi$ is a partial numbering of D_{re} which is equivalent to an admissible numbering of D_{re}.*

Proof. $\delta\circ\varphi$ is a partial mapping, since range $(\varphi)\not\subseteq\mathbb{F}$. Since δ satisfies (A1), there is a function $\Gamma\in C$ with range $\Gamma(f) = \{i \mid \beta(i) \prec \delta f\}$. So, for $\varphi_i\in\mathbb{F}$, range $\Gamma(\varphi_i)$ is a recursively enumerable set which means that $\delta\varphi_i \in D_{re}$.

Let η be any admissible numbering of D_{re}. Since range $\Gamma(\varphi_i)$ is r.e. and β applied to this set is directed, there is some $h\in R^{(1)}$ with range $\Gamma(\varphi_i) = W_{h(i)}$ and $\beta W_{h(i)}$ directed, so

$$\delta\varphi_i = \beta W_{h(i)} = \eta f(h(i)) \quad \text{for } f\in R^{(1)} \ (f \text{ from (Z2) for } \eta),$$

that is, $\delta \circ \varphi \leqslant \eta$ via $f \circ h \in R^{(1)}$. From (Z1) for η we conclude that there is some $g \in R^{(1)}$ with $\{i \mid \beta(i) < \eta(j)\} = \text{range } \varphi_{g(j)}$ where $\varphi_{g(j)} \in R^{(1)}$. Then we have $\eta(j) = \bigsqcup \beta \text{ range } \varphi_{g(j)} = \delta \Sigma \varphi_{g(j)}$ for $\Sigma \in C$ (Σ from (A2) for δ), and since $\Sigma \in C$, there is some function $k \in R^{(1)}$ with $\delta \Sigma \varphi_{g(j)} = \delta \varphi_{kg(j)}$, that means $\eta \leqslant \delta \circ \varphi$ via $k \circ g \in R^{(1)}$. □

There is even a straightforward way to obtain an admissible (total) numbering of D_{re} if an admissible representation of $\bar{D}$ is given: From recursion theory we know that there is some $h \in R^{(1)}$ with $\varphi_{h(i)} \in R^{(1)}$ and range $\varphi_{h(i)} = \text{Dom } \varphi_i \cup \{i_\perp\}$ for all i. Then the following holds.

Theorem 19. *If $\delta : \mathbb{F} \to D$ is an admissible representation, then $\eta : \mathbb{N} \to D_{re}$ with $\eta(i) := \delta \varphi_{h(i)}$ is an admissible numbering.*

Because of the similarity of the axioms for admissible representations and admissible numberings, the proof is easy. Thus, combining our admissible representations with φ, we obtain admissible numberings of the r.e. cpo elements for which a lot of recursion theoretical properties can be proved: For an effective cpo $\bar{D}$ let η be an admissible numbering of D_{re}. Then

- the recursion theorem holds for η, i.e., there is some $f_r \in R^{(1)}$ with $(\forall i)$ $\varphi_i \in R^{(1)} \Rightarrow \eta f_r(i) = \eta \varphi_i f_r(i))$.
- Rice's theorem holds for η, i.e., $\eta^{-1} X$ is not recursive whenever $\emptyset \subset X \subset D_{re}$.
- The theorem of Rice and Shapiro holds for η, i.e., for $A \subseteq D_{re}$, $\eta^{-1}(A)$ is r.e. iff there is an r.e.-open set $C \subseteq D$ with $A = C \cap D_{re}$, where r.e.-open means that C is the join of a r.e. set of open sets from the topological base of τ_D.

Furthermore, for two effective cpo's $\bar{D}_1$ and $\bar{D}_2$ with admissible representations η_1, η_2 a general version of the theorem of Myhill and Shepherdson has been proved in [19].

A theory of topologically admissible representations $\delta : \mathbb{F} \to D$ can be obtained by generalizing our theory of 'admissible' representations.

Let $\bar{D} = (D, \sqsubseteq, \perp, \beta)$ be a continuous cpo where β is an arbitrary numbering of a basis of $\bar{D}$. We do no longer require that $\{(i,j) \mid \beta(i) < \beta(j)\}$ is recursively enumerable. The proof of Theorem 8 yields a continuous (not necessarily computable) single-valuedness operator Γ_s. Define δ_m and δ_s as before. Lemma 10 still holds for this generalized situation. We substitute axioms (A1) and (A2) by 'continuous' versions:

(A1') There is a continuous function $\Delta : \mathbb{F} \to \mathbb{F}$ with range $\Delta(f) = \{i \mid \beta(i) < (f)\}$ for any $f \in \mathbb{F}$.

(A2') There is a continuous function $\Sigma : \mathbb{F} \to \mathbb{F}$ with $\delta \Sigma(f) = \bigsqcup \beta \text{ range } (f)$ whenever β range (f) is directed for any $f \in \mathbb{F}$.

We call a representation $\delta : \mathbb{F} \to D$ 't-admissible', iff it satisfies (A1') and (A2'). The proof of Theorem 12 yields that any standard representation is t-admissible. We generalize Definition 13: $\delta_1 \leq_t \delta_2$, iff there is some *continuous* $\Omega:\mathbb{F}\to\mathbb{F}$ with $(\forall f \in A_1)\delta_1(f)=\delta_2\Omega(f)$, and $\delta_1 \equiv_t \delta_2 :\Leftrightarrow (\delta_1 \leq_t \delta_2 \wedge \delta_2 \leq_t \delta_1)$. Then the corresponding topological versions of Lemma 14, Corollaries 15 and 16 and Theorem 17 hold. By Lemma 20, property (A1') expresses continuity.

Lemma 20. *δ satisfies* (A1') $\Leftrightarrow$ *δ is continuous.*

Proof. Let δ satisfy (A1'), that is, by Lemma 14: $(\exists\Delta, \Delta$ continuous$)$ $\delta=\delta_m\Delta$. Then δ is continuous since δ_m is continuous (Lemma 10).

Suppose δ is continuous. Then

$$\beta(i)<\delta f \Leftrightarrow \delta f\in O_{\beta(i)} \Leftrightarrow (\exists n)\delta[f^{[n]}]\subseteq O_{\beta(i)}.$$

Define $\Theta:\mathbb{F}\to\mathbb{F}$ by

$$\Theta(f)(\langle n,i\rangle):=\begin{cases} i & \text{if } \delta[f^{[n]}]\subseteq O_{\beta(i)},\\ i_\perp & \text{otherwise.}\end{cases}$$

Then Θ is the continuous function claimed in (A1'). $\square$

Property (A2') has another interesting topological consequence.

Lemma 21. *δ satisfies* (A2') $\Rightarrow$ $(\forall M\subseteq D)(\delta^{-1}M$ *open* $\Rightarrow$ M *open*).

Proof. Property (A2') immediately implies $\delta_m=(\delta\Sigma)|_{\mathrm{MON}}$. Therefore

$$\delta_m^{-1}(M)=(\delta\Sigma)^{-1}(M)\cap \mathrm{MON}=\Sigma^{-1}(\delta^{-1}(M))\cap\mathrm{MON} \quad \text{for any } M\subseteq D.$$

Suppose $\delta^{-1}(M)$ is open, then $\delta_m^{-1}(M)$ is open. Thus, by Lemma 10, M is open. $\square$

As a direct consequence we get the following theorem.

Theorem 22. *Let δ be an* t-*admissible representation of a continuous cpo $\bar{D}$. Then*

$$(\forall M\subseteq D)(M \textit{ open} \Leftrightarrow \delta^{-1}(M) \textit{ open}).$$

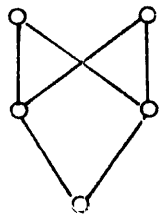
This means that τ_D is the quotient topology relative to δ and Baire's topology τ_B, if δ satisfies (A1') and (A2').

Note, that this is especially true for an admissible representation δ of an effective cpo $\bar{D}$.

All of these properties indicate that our definition of admissible representations is a very natural one. Especially Lemmas 20 and 21 shed some new light on the utm-theorem and the smn-theorem for 'effective Gödel numberings' of $P^{(1)}$ [12].

As already announced we shall prove that a corresponding theory does not exist with P_ω instead of $\mathbb{F}$ as set of names. Computability on P_ω can be defined by enumeration operators (Rogers [12]). Any computable $\Gamma:P_\omega\to P_\omega$ is continuous and especially monotone.

Let $\bar{D} = (D, \sqsubseteq, \perp, \beta)$ be some effective cpo, such that $\bigsqcup X$ does not exist for some finite set $X \subseteq D$. For example the following cpo with appropriate numbering of the basis can be chosen:



For $x \in D$ define $I_x := \{i \mid \beta(i) < x\}$. Define

$$\mathrm{IN} \subseteq P_\omega \text{ by } \mathrm{IN} := \{I_x \mid x \in D\} \quad \text{and} \quad \delta_{\mathrm{m}} : \mathrm{IN} \to D \text{ by } \delta_{\mathrm{m}}(I_x) := \bigsqcup \beta I_x.$$

Then δ_{m} is bijective, and $A \subseteq B \Leftrightarrow \delta_{\mathrm{m}}(A) \sqsubseteq \delta_{\mathrm{m}}(B)$ for all $A, B \in \mathrm{IN}$. Suppose, there is some arbitrary $\delta : P_\omega \to D$ (onto) such that $\delta = \delta_{\mathrm{m}} \Gamma$ and $\delta_{\mathrm{m}} = \delta \Sigma$ for monotone functions $\Gamma, \Sigma : P_\omega \to P_\omega$. Then $\delta_{\mathrm{m}} \Gamma \cup \Sigma \delta_{\mathrm{m}}^{-1}(X)$ has the property of the least upper bound of X, a contradiction.

Therefore, there are many relevant cpo's for which even t-admissible representations $\delta : P_\omega \to D$ do not exist.

5. Characterizations of continuous and computable functions on cpo's

We consider the following diagram, where $\bar{D}_1$ and $\bar{D}_2$ are effective cpo's:

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "\Gamma"] \arrow[d, "\delta_1"'] & \mathbb{F} \arrow[d, dashed, "\delta_2"] \\
D_1 \arrow[r, "p"'] & D_2
\end{tikzcd}$$

We assume that δ_1 and δ_2 are surjective. The diagram commutes, iff $p\delta_1 = \delta_2 \Gamma$. In this case it can be interpreted as follows: function p is 'computed' by function Γ via the representations δ_1 and δ_2. Obviously, for every p there is some Γ such that $p\delta_1 = \delta_2 \Gamma$. On the other hand, Γ determines some p, iff Γ is (δ_1, δ_2)-extensional, i.e., $(\delta_1 g = \delta_1 h \Rightarrow \delta_2 \Gamma g = \delta_2 \Gamma h)$.

The following questions arise. Does every continuous p correspond to a continuous extensional Γ and vice versa? Does every computable p correspond to a computable Γ and vice versa? We shall answer these questions positively under the assumption that δ_1 and δ_2 satisfy (A1′) and (A2′) or (A1) and (A2), respectively.

Theorem 23. *Let $\bar{D}_i$ be effective cpo's, let $\delta_i : \mathbb{F} \to D_i$ be representations for $\bar{D}_i$ $(i = 1, 2)$.*

(1) *Suppose δ_1 satisfies* (A1) *and δ_2 satisfies* (A2). *Then for any $(\bar{D}_1, \bar{D}_2)$-computable p there is some computable $\Gamma : \mathbb{F} \to \mathbb{F}$ with $p\delta_1 = \delta_2\Gamma$.*

(2) *Suppose δ_1 satisfies* (A2) *and δ_2 satisfies* (A1). *Then every (δ_1, δ_2)-extensional computable $\Gamma : \mathbb{F} \to \mathbb{F}$ determines a $(\bar{D}_1, \bar{D}_2)$-computable function p with $p\delta_1 = \delta_2\Gamma$.*

Proof. (1) Suppose, p is computable. Then

$$\mathrm{range}(g) = \{\langle i, j\rangle \mid \beta_2(j) \prec p\beta_1(i)\} \quad \text{for some } g \in R^{(1)}.$$

δ_1 satisfies (A1), i.e.,

$$(\exists \Delta \in C)\beta_1(i) \prec \delta_1(f) \Leftrightarrow \Delta(i \oplus f) \neq \lambda x.0 \quad \text{for } f \in \mathbb{F}.$$

Define $\Delta' : \mathbb{F} \to \mathbb{F}$ by

$$\Delta'(i \oplus f)(j) := \begin{cases} 1 & \text{if } (\exists k \leq \pi_1(j))(\exists l \leq \pi_2(j)) \\ & \quad \Delta(k \oplus f)(l) \neq 0 \wedge \langle k, i\rangle \in \{g(0) \cdots g(j)\}, \\ 0 & \quad \text{otherwise.} \end{cases}$$

Then $\Delta' \in C$ and

$$\begin{aligned} \Delta'(i \oplus f) \neq \lambda x.0 &\Leftrightarrow (\exists k)(\exists l)(\Delta(k \oplus f)(l) \neq 0 \wedge \langle k, i\rangle \in \mathrm{range}(g)) \\ &\Leftrightarrow (\exists k)(\Delta(k \oplus f) \neq \lambda x.0 \wedge \beta_2(i) \prec p\beta_1(k)) \\ &\Leftrightarrow (\exists k)(\beta_1(k) \prec \delta_1 f \wedge \beta_2(i) \prec p\beta_1(k)) \\ &\Leftrightarrow \beta_2(i) \prec p\delta_1(f) \quad \text{(by monotony of } p \text{ and Lemma 1(4)).} \end{aligned}$$

Thus, $p \circ \delta_1$ satisfies (A1). By Corollary 15 we conclude $p \circ \delta_1 = \delta_2\Gamma$ for some $\Gamma \in C$.

(2) Let $\Gamma \in C$ be extensional and $p\delta_1 = \delta_2\Gamma$. Then, since δ_2 and Γ are continuous, $p\delta_1$ is a continuous function. Since δ_1 satisfies the assumption of Lemma 21, we can conclude that p is continuous.

Finally we have

$$\begin{aligned} p\beta_1(i) &= p\delta_1\Sigma_1(\lambda x.i) \quad \text{(by (A2) for } \delta_1) \\ &= \delta_2\Gamma\Sigma_1(\lambda x.i). \end{aligned}$$

By (A1) for δ_2 we have

$$\begin{aligned} \beta_2(j) \prec p\beta_1(i) &\Leftrightarrow \beta_2(j) \prec \delta_2\Gamma\Sigma_1(\lambda x.i) \\ &\Leftrightarrow j \in \mathrm{range}\, \Delta_2(\Gamma\Sigma_1(\lambda x.i)). \end{aligned}$$

Since Δ_2, Γ and Σ_1 are computable, $\{\langle i, j\rangle \mid \beta_2(j) \prec p\beta_1(i)\}$ is recursively enumerable, and therefore p is computable. □

Definition 24. *Let δ_i be a representation for $\bar{D}_i$ $(i = 1, 2)$, and let $p: D_1 \to D_2$. Then p is called (δ_1, δ_2)-computable, iff $p\delta_1 = \delta_2\Gamma$ for some computable $\Gamma \in C$.*

With this definition we can formulate the main result of this section.

Theorem 25. *Let δ_i be an admissible representation of the effective cpo $\bar{D}_i$ $(i = 1, 2)$ and let $p: D_1 \to D_2$ be a function. Then p is computable, iff p is (δ_1, δ_2)-computable.*

This means, using admissible representations, that cpo-computable functions are exactly computed by the extensional computable functions on names.

There are again weaker versions of these results for the case of 'continuity' instead of computability. The proofs are the same or even easier.

6. Conclusion

We have defined admissible representations $\delta: \mathbb{F} \to D$ for effective cpo's and motivated this definition in several ways. We have proved that the computable functions between cpo's are exactly the (δ_1, δ_2)-computable functions for admissible representations. Our definition generalizes the definition of admissible representations $\gamma: A \to \mathbb{R}$ (real numbers) with $A \subseteq \mathbb{F}$ which are used in recursive analysis (e.g., see Hauck [6]).

Let $\bar{D}$ be the effective cpo of closed intervals on $\mathbb{R}$ defined in [19]. Then w.l.o.g. $\mathbb{R} \subseteq D$. If δ is a cpo-admissible representation, then $\delta|_A$, $A = \delta^{-1}\mathbb{R}$, is admissible for the purpose of recursive analysis.

Finally we have seen that combining admissible representations with a standard numbering φ of $\mathbb{P}^{(1)}$ yields the admissible numberings of D_{re}, that means that admissible representations are a natural generalization of admissible numberings.

The results obtained so far assure that $\mathbb{F}$ is a reasonable set of names and that admissible representations are exactly the 'natural' or 'effective' ones.

As a next step computations on $\mathbb{F}$ should be studied and a theory of computational complexity for cpo-functions should be developed.

Such a theory would have important applications in (recursive) analysis.

References

[1] G. Asser, Rekursive Wortfunktionen, *Zeitschrift für Math. Logik und Grundlagen der Math.* **6** (1960) 258–278.

[2] M. Davis, *Computability and Unsolvability* (McGraw-Hill, New York, 1958).

[3] H. Egli and R. L. Constable, Computability concepts for programming language semantics, *Theoret. Comput. Sci.* **2** (1976) 133–145.

[4] Yu.L. Eršov, Theorie der Numerierungen I, *Zeitschrift für Math. Logik und Grundlagen der Math.* **19** (1973) 289–388.

[5] G. Gierz, K.H. Hoffmann, K. Keimel, J.D. Lawson, M. Mislovc and D.S. Scott, *A Compendium of Continuous Lattices* (Springer, Berlin, 1980).
[6] J. Hauck, Berechenbare reelle Funktionen, *Zeitschrift für Math. Logik und Grundlagen der Math.* **19** (1973) 121–140.
[7] A. Kanda and D. Park, When are two effectively given domains identical, in: K. Weihrauch, ed., *Theoretical Computer Science 4th GI Conf.*, Lecture Notes in Comp. Sci. **67** (Springer, Berlin, 1979) pp. 170–181.
[8] K. Kuratowski, *Topology* (Academic Press, New York-London, 1966).
[9] A.I. Mal'cev, *Algorithms and Recursive Functions* (Wolters-Noordhoff, Groningen, 1970).
[10] G. Plotkin, T^{ω} as a universal domain, *J. CSS* **17** (1978) 209–236.
[11] A. Reiser and K. Weihrauch, Natural numberings and generalized computability, *EIK* **16** (1980) 1–3; 11–20.
[12] H. Rogers, *Theory of Recursive Functions and Effective Computability* (McGraw-Hill, New York, 1967).
[13] H. Rogers, Gödelnumberings of partial recursive functions, *JSL* **23** (1958) 331–341.
[14] E. Sciore and A. Tang, Admissible coherent cpo's, in: G. Ausiello and C. Böhm, eds. *Automata, Languages and Programming, 5th Colloquium*, Lecture Notes in Comp. Sci. **62** (Springer, Berlin, 1978) pp. 440–456.
[15] D. Scott, Continuous lattices, in: F. W. Lawvere, ed., *Toposes, Algebraic Geometry and Logic*, Lecture Notes in Mathematics **274** (Springer, Berlin, 1972) pp. 97–136.
[16] D. Scott, Data types as lattices, *SIAM J. Comput.* **5** (1976) 522–587.
[17] M.B. Smyth, Effectively given domains, *Theoret. Comput. Sci.* **5** (1977) 257–274.
[18] K. Weihrauch, Recursion and complexity theory on cpo's, in: P. Deussen, ed., *Theoretical Computer Science, 5th GI Conf. 1981*, Lecture Notes in Comp. Sci. **104** (Springer, Berlin, 1981) pp. 195–202.
[19] K. Weihrauch and T. Deil, Berechenbarkeit auf cpo's, *Informatik-Berichte* **63** (RWTH, Aachen, 1980).
[20] K. Weihrauch and G. Schäfer, Admissible representations of effective cpo's, in: J. Gruska and M. Chytil, ed., *Mathematical Foundations of Computer Science 1981*, Lecture Notes in Comp. Sci. **118** Springer, Berlin, 1981) pp. 544–553.